L'héritage de Kiyosi Itô

What Makes Kiyosi Itô Famous on Trading Floors?

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Nicole El Karoui

UPMC/Ecole Polytechnique, Paris elkaroui@gmail.com

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Plan

1 My own experience

2 Quantitative finance

3 Stochastic Processes

4 Constrained BSDEs

What makes Kiyosi Itô famous on trading floors?

1. A brief historical overview of mathematical finance
2. The role of derivative markets and the daily risk-management
3. Calibration issues and No-arbitrage bounds in classical case:

   Via Skohorod embedding problem, or Optimal transportation theory
4. Hedging with constraints: Unified point of view via BSDEs:

   Theory and Numerical Applications
5. Crisis induces new priorities in research on global system:
   - Liquidity constraints and counterparty risk at the level of the bank
   - Contagion and systemic risk: Mean field models, Networks
6. Concluding remarks

My first contact with Japanese Probabilities

My first contact with Markov processes

- In May 1968, I was a student in a Master's program in probability at IHP with the Professor Neveu, Marie Duflo...
- I explained my interest for stochastic processes, and in a PhD thesis in this domain.
- Outside, it was the "May 1968 events", very animated, with battle between police and students, because IHP is very close to the Sorbonne. It was "surreal" to discuss my PhD thesis in this context.
-

He said to me "why not"?

I had just received a very interesting paper from Japan. You can read it, and come back to clarify the subject of the PhD thesis.

Sweeping-out of functional additive, Motoo(1965)

THE SWEEPING-OUT OF ADDITIVE FUNCTIONALS AND PROCESSES ON THE BOUNDARY

MINORU MOTOO

1. Introduction

In this paper we shall consider the sweeping-out of additive functionals in Markov processes, and its application to processes on the boundary (U-processes). In sections 3 and 4 we define the sweeping-out of additive functionals and investigate their properties. In section 5, we consider a special additive functional (i.e. a time additive functional) and its inverse function. In section 6, using the additive functional defined in section 5, we transform the original process by the time change, and obtain a certain form of the process on the boundary (U-process) introduced by T. Ueno [9]. We show that this process is sufficiently regular if the original one is so. This paper is an introductory part of the investigation of U-processes. More detailed arguments of U-processes and their application to the boundary value problems are treated in [8]. The author wishes to express his gratitude to Mr. K. Sato and Mr. T. Ueno for their helpful advices and encouragements.

It was a very interesting paper on Markov Processes, extending in some sense Itô 's result on the decomposition of Markov Processes around a point.
But, I was very ignorant of the Markov process theory.

- Finally, we organized a working group to read the paper, and six years later we proposed an extension to "general Markov processes" (Asterisque, 1975)
- My admiration for the Probabilistic group in Japan continues to be great.

My debt to the Japanese School, as a student of Neveu and P.A. Meyer (1968...)

- Working on the paths, using time translation, change of time, killing operator..) and taking expectations only at the end.
- Recurrent message of P.A Meyer, still in reference to Pr. Itô, but not obvious in Markov theory.
- Semimartingale and stochastic integral.

Louis Bachelier

- In 1900, a young mathematician, Louis Bachelier defended his PhD thesis before a jury whose chairman was Henri Poincaré.
- The thesis title, published in the “Annales de l’Ecole Normale Supérieure” , was *Théorie de la spéculation.*
- Poincaré reported: original but it is a pity that it concerns financial markets

He wrote this very enigmatic sentence :

> *Although we will probably never predict stock price movements reliably, however it is possible, to study the static state of the market, that is to establish the law of probability for the variations of the stocks accepted at this moment by the market.*

Axiomatic for continuous time pricing problem

(i) Put an axiomatic for continuous time finance.

(ii) Based on the time consistency of prices of derivatives.

(iii) Deduced (with some approximation) that prices satisfy the heat equation.

(iv) Then, introduce Brownian motion as a limit of a random walk.

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Quantitative Finance: Historical Overview

1970-1974: Deregulation versus Financial Innovation

- United States' decision to float the dollar 15/08/1971 (Nixon) Great
- monetary disorder
- Financial Innovation: Markets for Future and Options Contracts
  - Chicago Board of Options Exchange opens in 1973.
  - Options become financial instruments with which risk can be managed.

- 1900: Bachelier defends his thesis on Theory of Speculation.
- 1960-70: Portfolio Theory: Markowitz.
- 1973 : Black-Scholes-Merton theory of option pricing and hedging portfolio.

Future Exchanges

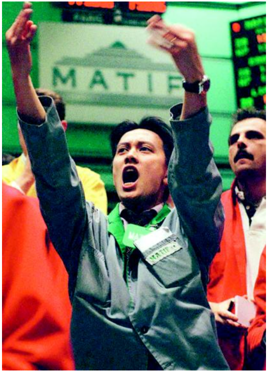
Definition

- Forward contracts, which obligated one counterparty to buy and the other to sell a fixed amount of securities at an agreed date in the future T at a fixed price today.
- Futures contracts are the standardized version of forward contracts by clearinghouses, or for collateralized transactions.
- Option Contract: the right but not the obligation, to buy (sell) something in the future at a given price called = exercise price = strike price = K, often closed to the forward price.

Use

- as protection against fluctuations and large movements on the market.
- easy instruments for speculation (with anticipation on the future evolution of the underlying).

The MATIF (1986)



(i) The first French futures Market, the MONEP in 1987.

(ii) Major French banks anticipated the event.

(iii) A sophisticated, very quantitative activity.

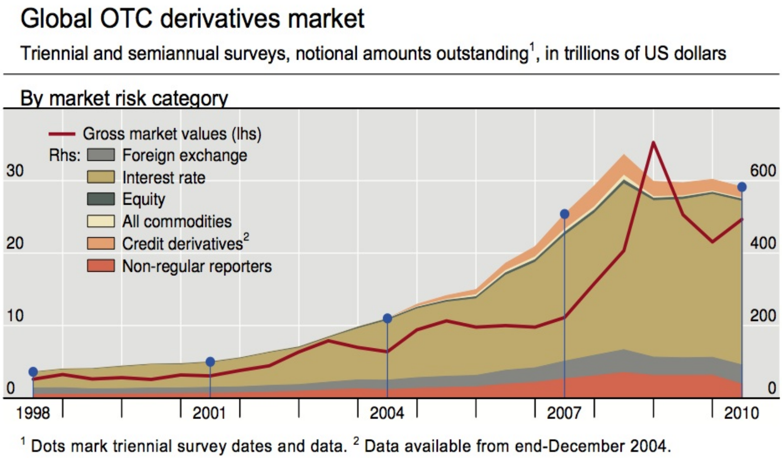
Golden Age of the Financial Industry: 1995 - 2008

Golden Age of Financial Innovation

- After 2003, Boom of the derivatives market, with non-tradable underlying: (Volatility, Credit, Subprimes)
- "Shadow Banking": Hedge funds and high-frequency trading
- Banking, investment and finance become a quantitative and data-driven industry.

Golden Age of Quantitative Finance

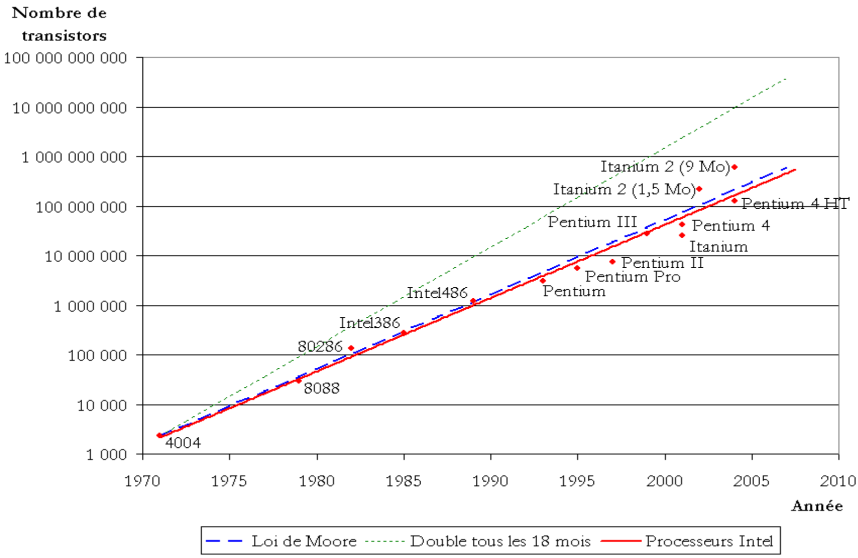
- Thousands of scientists, engineers and mathematicians enter the field.
- More that 70 top universities have degree programs in Financial Mathematics and Engineering.
- Research publications on mathematical problems in investment and finance increase dramatically.



ure: Evolution of notional size of OTC derivatives market. Source: BIS.

Market Derivatives 1998-2010,Bis, in Trillions

Exponential Growth in Computing Power : Moore Law



Large Depression of Financial Industry

- 2007-2008 Credit Crunch/ Lehman Collapse
  - The excesses of the finance industry was dragging down the whole economy.
  - Credit crunch was based on subprime risks, a lowering of underwriting standards that drew people into mortgages.
  - Diffusion of the home mortgage crisis in any financial place through securitization via MBS
  - Mortgage-backed securities (MBS) depend on the performance of hundreds of mortgages.
- Drastic reduction of credit derivatives business
- Liquidity crisis in the interbank market

Quantitative Finance: Three Pillars

Practice

- Financial innovation
- Pricing
- Risk management

Mathematics

- Continuous Time Finance
- Stochastic Calculus and PDEs
- Optimization

Numerical implementation

- Modelling and Computing (Monte Carlo)
- Calibration
- Risk management in Practice/Regulation /New Challenge

Program of the Master Degree PVI-X

I SURVIVED...DEA EL KAROUI, Year 2010/ Master's Program started in 1990

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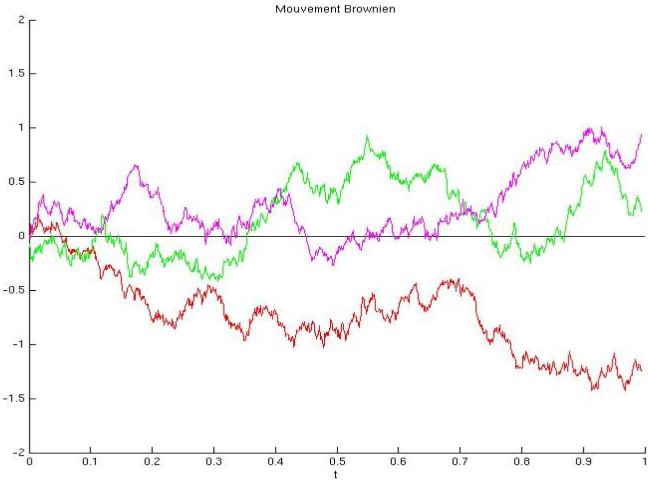
Examples of financial paths

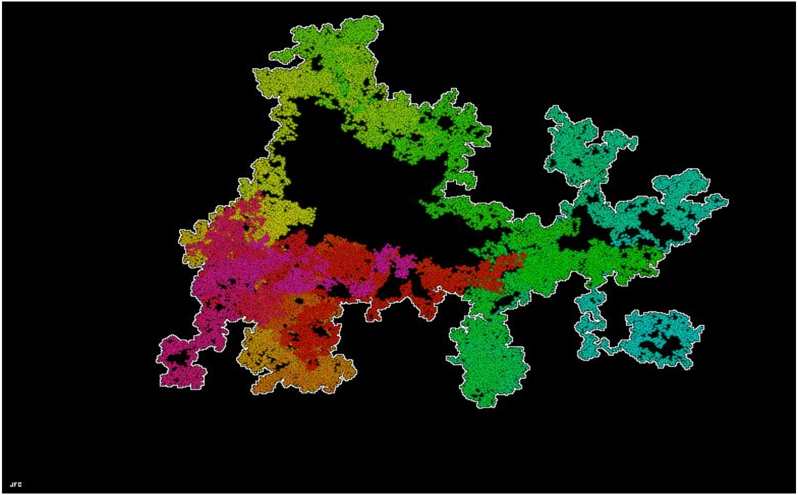
CAC 40 and FTSE between 1996 and 2008

Brownian motion simulation

Simulated path of Brownian motion with different diffusion coefficients



Two-dimensional Brownian, Colonna



Stochastic Process Theory

From observation to trajectories

During these 70 years, the mathematical theory of Brownian motion and stochastic processes greatly expanded with the main contributions coming from Japanese and French mathematicians.

- Einstein (1905) "observed" and introduced the heat equation
- Wiener (1913) used mathematics of signal theory
- Paul Levy (1930), introduced the PAI
- Kiyosi Itô (1940)....comes back from the PDE to the paths
- Kolmogorov (1930)

In honor of Professor Kiyosi Itô

- The mathematical concepts have essentially been developed in the last 40-50 years with purely mathematical motivation:
  - stochastic integration
  - stochastic differential calculus
  - Itô's formula
- Professor Itô reintroduced the "paths" in the center of the theory. It is not enough to have an estimate of the distribution, of the expected value of some risky quantity, even given by a fine calculation via PDE.
- 50 years after, this point finds an exact translation in finance with the theory of hedging portfolio (Merton).
- The Japanese School of Stochastic Processes is still one of the best in the world.

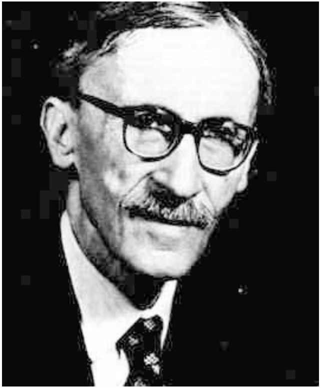
Stochastic calculus: Itô (1936-40)

What is it? The objective is to define integral and differential calculus for non-differentiable functions.

- Even if it is still possible to give meaning to $\sum_{i=0}^{n-1} \delta_{t_i}(X_{t_{i+1}} - X_{t_i}) = V_{t_n}$, it is more subtle to give meaning to $V_t = \int_0^t \delta_u \, dX_u$,
- and to represent $f(V_t)$ using similar integrals (differential calculus).
- These objects were the theoretical tools used by finance theory in 1970. Since

  If X_t is the asset price at time t
- V_{t_n} is the gain process associated with portfolio strategy with δ_{t_i} risky asset at time t_i. The integral has the same meaning for continuously traded strategy.

Professors Paul Levy and Kiyosi Itô



Pricing and Hedging Problems from BS

- Pricing rule
  - The price of an option contract is the cost of the hedge
  - Hedging strategy is based on dynamic self-financing portfolio V, written on the tradable asset X with value at T closed to the exposure $V_T \sim h(X_T) = (X_T - K)^+$
- New paradigm in risk management
  - The problem is not to estimate the expected losses
  - Future time is used as a tool for "diversification"
- Operational Constraints
  - The underlying of the contract is tradable on the market
  - Small trade with non-impact on the price of the underlying
  - Liquidity and weak transaction cost.

Liquidity is the possibility to trade positions without generating market instability

Black and Scholes Solution

Self-financing portfolio on tradable underlying X

- The variation of cumulative gain process V_t of a trading strategy, with δ_t shares held at time t is
  - the gain due to the risky investment $\delta_t dX_t$
  - the interest (short rate r_t) due to the residual wealth $V_t - \delta_t X_t$.
- A self-financing hedging equation

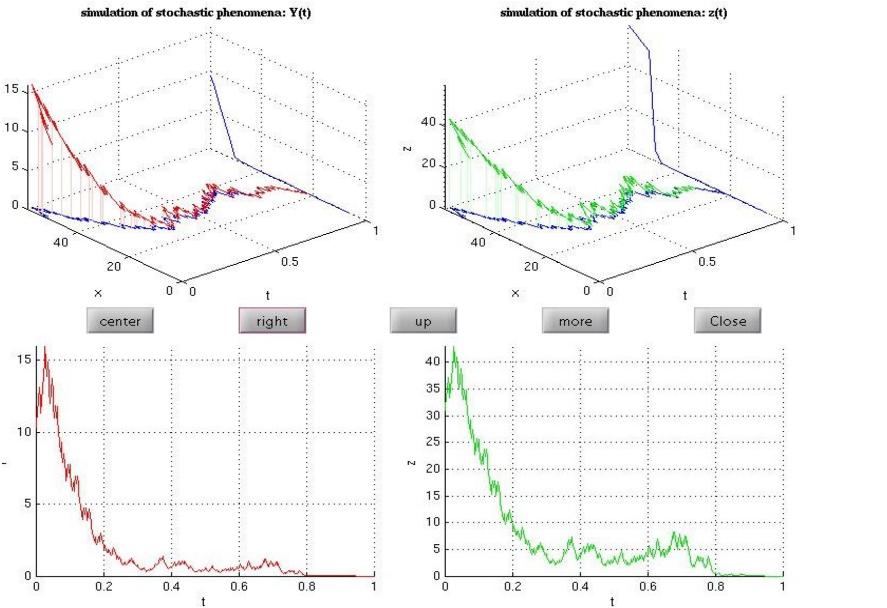
$$\begin{cases} dV_t = r_t(V_t - \delta_t X_t)dt + \delta_t dX_t = r_t V_t dt + \delta_t(dX_t - r_t X_t\, dt), \\ V_T = (X_T - K)^+ \quad \text{terminal constraint} \end{cases}$$

BS Solution for GBM $dX_t = X_t[r\,dt + \sigma(dW_t + \theta dt)]$, $\mu = r + \theta\sigma$

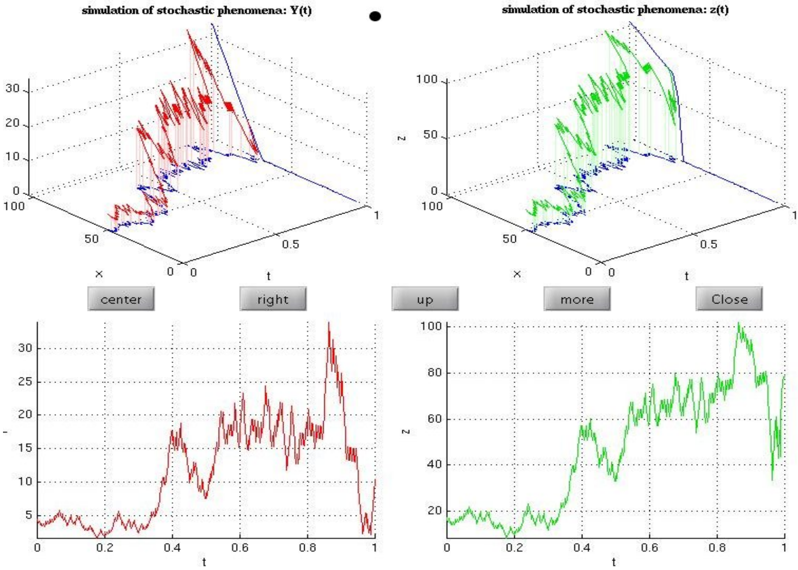
- BS Formula for Call Option: no dependence in the trend θ
  $C^{BS}(t, x, r, K, T, \sigma) = x\,N(d_1) - Ke^{-r(T-t)}\,N(d_0)$
- Hedging strategy: $\delta_t = \partial_x C^{BS}(t, x, r, K, T, \sigma) = N(d_1)(\equiv 0.55)$

The curse of the derivative as an interpretation of the δ

Call(50,50): Hedging portfolio of Call (blue = asset path, red = portfolio value, green = portfolio's risky part)



Call(50,70): Hedging portfolio of Call



Calibration Issues: Implied Volatility

Liquid Markets: Exchange Markets, Currencies....

First period: 1973-1987

- quoted option prices are available on the market
- Hedging rule: One price, one implied volatility, one hedge
- Used several times a day at hedging times when market moves

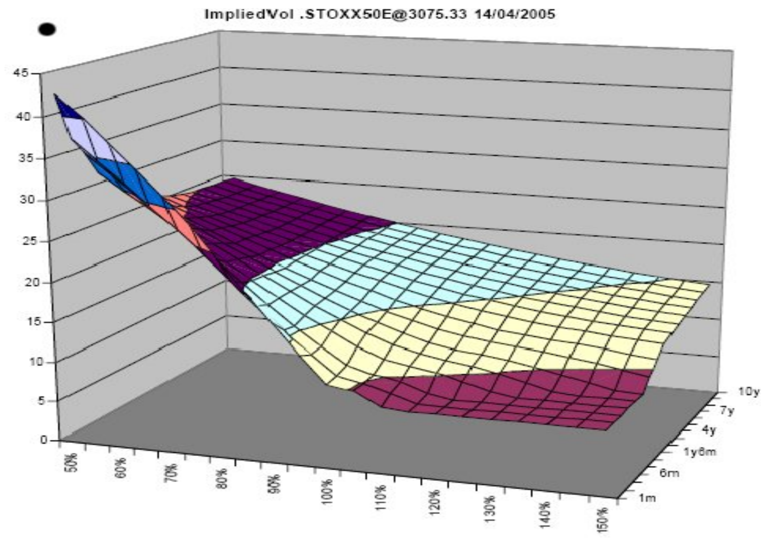
Second period: 1993—

- More complex derivatives depending on volatility
- Liquid options contracts are used as hedging instruments
- but No flat Implied volatility surface

cbleu Quantitative formulation

- Implied Volatility $\Sigma^{imp}(T,K)$ from quoted option prices $C^{obs}(T,K)$ is defined by $C^{obs}(T,K) = C^{BS}(t_0, x_0, T, K, \Sigma^{imp}(T,K))$
- Implied hedging strategy $\Delta^{imp}_{t_0,x_0}(T,K) = \partial_x C^{BS}(t_0, x_0, T, K, \Sigma^{imp}(T,K))$

Implied Volatility Surface/SP500



Implied Diffusion

Liquid Markets: The Data set is the family of Call prices for every (T and K)

- First, verify the coherence of the data (convex, $\downarrow K$, and $\uparrow T$ (if $r = 0$))
- The aim is to price and hedge path dependent options derivatives (barrier options, Asian, Lookback) with these liquid vanilla options, with calibrated Markovian model

Mathematical Issues (Dupire (1996))

- Local volatility and Dual PDE (Dupire (1996)). There exists a function $\sigma^{loc}_{t_0,x_0}(T, K)$ given by the dual PDE
$$\frac{\partial C}{\partial T} = \frac{1}{2}(\sigma^{loc}_{t_0,x_0})^2(T,K)K^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}(T,K)$$
- The forward start call prices $C_{t_0,x_0}(t, x, T, K) = v(t, x)$ is the solution to the backward PDE, with terminal condition $(x - k)^+$
$$v'_t(t,x) + \frac{1}{2}(\sigma^{loc}_{t_0,x_0})^2(t,x)x^2 v_{xx}(t,x) + rxv_x(t,x) - rv(t,x) = 0$$

Implied Diffusion, II

Mathematical references

- Dupire.B and alli: Formally, $(\sigma^{loc}_{t_0,x_0})^2(T,K) = \mathbb{E}[\tilde{\sigma}^2_t | S_T = K]$
- Gyongy, I. (1986), *Mimicking the one-dimensional marginal distributions of processes having an Itô differential*
- Brunick, Shreve (2011), *Mimicking an Itô Process by a SDE with same marginals: Multi-dimensional case with very weak assumptions ($\Longrightarrow$ non uniqueness*

Drawbacks of the method

- In the market, a finite number of option prices are available, leading to an ill-posed inverse problem. Intensively studied from a PDE point of view, (penalization, and other methods)
- Do not forget the constraints of the speed of excution (less than 1mn), and recalibration techniques.
- Often used in currency markets, as convenient for pricing and hedging barrier options.

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Constraint portfolios via BSDE's

Unified point of view First introduced by Peng and Pardoux in 1990.

Definition (Backward Stochastic Differential Equation)

BSDE is a stochastic differential equation of the form

$$-dY_t = f(\omega, t, Y_t, Z_t)dt - Z_t\, dW_t, \quad Y_T = H, \text{where}$$

- the random variable H is called the TERMINAL CONDITION
- the random function $f(\omega, t, y, z)$ is called the COEFFICIENT OR DRIVER
- A solution is a pair (Y, Z) of adapted processes such that the previous equation holds. True in the unif Lipschitz case.
- In the Markovian case, where $f(t, X_t, y, z)$ and $H_T = g(X_T)$, the solution is a function $V(t, X_t)$, where $V(t, x)$ is the solution of the non-linear PDE with generator L^X of the diffusion process X

$$V_t(t,x) + L^X v(t,x) + f(t,x,v(t,x),\nabla V(t,x)) = 0,\, V_T(t,x) = g(x)$$

Different Points of view

- Financial interpretation:
  - The target is H
  - The strategic processes (portfolios) have constraint forward dynamics driven by (V_0, Z_t)

$$dV_t = -f(\omega, t, V_t, Z_t)dt - Z_t\, dW_t, \quad V_0 \text{ given}$$

  with eventually more complex constraints on Z (Bouchard (2010)).

- Superhedging problem If no solution exists, we try to solve the pb via the notion of super solution or (superhedging), defined at time 0 as
  - the "minimum" of Y_0 such that there exists an admissible portfolio $(V_t), \quad V_0 \geq Y_0$ such that $V_T \geq H$.

Exemples

- From the crisis, funding with collateral
  - Different bilateral agreement often $C_t = \phi(V_t)$ is a convex function of the transaction.
  - Different funding interest rate: r_t^f is **stochastic** funding interest rate, r_t^c is the **stochastic** collate interest rate.
- Forward equation $dV_t^{csa} = [r_t^c C_t + r_t^f (V_t^{csa} - C_t]dt + \delta_t(dX_t - r_t^f X_t dt)$
- BSDE coefficient $(C = (V^{csa})^+)$:

$$-f(t, y, z) = r_t^c y^- + r_t^f y^+ z.\theta_t^f = (r_t^c - r_t^f)y^- + r_t^f y + z.\theta_t^f$$ Partial

- Hedging Only few assets are available for the hedging. Denoted by I_K the convex indicator functions $= 0$ in K_t, ∞ if not, and by I_K^n a linear growth regulation. I_K^n is added to the standard coefficient

BSDE and Optimization

Assume a Brownian filtration. Linear BSDE

- For any bounded β_t, γ_t, $\phi \in \mathbb{H}^2$, there exists a unique $\mathbb{H}^2$ solution of the LBSDE, $-dY_t = [\phi_t + Y_t\beta + Z_t\gamma_t]dt - Z_t dW_t, \quad Y_T = \xi \in \mathbb{L}^2$,
- Pricing Rule $Y_t = \mathbb{E}\big[\xi H_t^T + \int_t^T H_t^s \phi_s ds | \mathcal{F}_t\big]$, where

$$H_t^s(\beta,\gamma) = \exp\Big(\int_t^s \beta_r dr\Big)\exp\Big(\int_t^s \gamma_r^\top dW_r - 1/2\int_t^s |\gamma_r|^2 dr\Big)$$

  the adjoint process (or state price density in finance).
- Comparison theorem: $\xi \geq 0$ and $\phi_t = f(t,0,0) \geq 0$ implies $Y_t \geq 0$

Optimization

- Value function of the optimization problem with convex generator $f(t,y,z) = sup\{\beta_t y - \gamma_t z - \alpha_t(\beta,\gamma) | r_t \in B_t, \theta_t \in K_t\}$.

$$Y_t(\xi) \quad = \quad \text{" sup"}_{r_t \in H_t, \theta_t \in K_t} \mathbb{E}\big[\xi\, H_t^T(\beta,\gamma) + \int_t^T H_t^s(\beta,\gamma)\alpha_s(\beta,\gamma) ds | \mathcal{F}_t\big]$$

Utility Risk Measure, and g expectation

The skeptic gamer, (G. Shaffer) = Coherent Risk Measure

- The operator $\xi \mapsto Y_t(\xi_T)$ is increasing, concave if g is concave, and defines a dynamic "Backward Stochastic Utility", using eventually $u(\xi_T)$ as the terminal condition. (g-expectation Peng (1995), recursive utility generalization, Duffie-Epstein (1992))

Sup-convolution

- Given two concave increasing coefficients g^A and g^B, the BSDE with coefficient $g^A \square g^B$, (if it is not still $+\infty$), $(g^A \square g^B)(y,z) = \sup_{((y^A,z^A),(y^B,z^B))}[g^A(y^A,z^A) + g^B(y^B,z^B) | y^A + y^B = y, z^A + z^B = z)$
- is the sup-convolution of Y^A and Y^B, $(Y^A \square Y^B)_t(\xi_T) = \sup_{(\xi^A,\xi^B)}[Y^A_t(\xi^A) + Y^B_t(\xi^B) | \xi^A + \xi^B = \xi_T)]$

Indifference pricing, and other extensions

Indifference pricing

- Given a family of admissible linear portfolios $V_t^\pi(x)$, the indifference price of ξ_T, if there exists, is defined as the minimum of the wealth p such that

$$\inf_p \max_\pi Y_t(V_T^\pi(x+p)-\xi_T)=\max_\pi Y_t(V_T^\pi(x))$$

- associated with a cash monetary risk measure

Quadratic BSDE: application to entropic risk measure

Numerical methods for BSDEs

The most challenging field in the domain

- With spectacular results in ten years
- Multilevel Monte Carlo
- Non-linear methods are not efficient in the aggregation process
- Very strategic in risk management to calculate the risk indicators

A few remarks thanks to E.Gobet.

Our aim

- to simulate Y and Z
- to estimate the error, in order to tune finely the convergence parameters.

Numerical methods for BSDEs

The BSDE language is well-adapted to the simulation

It is quite intricate and demanding since

- it is a non-linear problem (the current process dynamics depend on the future evolution of the solution).
- it involves various deterministic and probabilistic tools (also from statistics).
- the estimation of the convergence rate is not easy because of the non-linearity of the loss of independence (mixing of independent simulations).

The BSDE case

The BSDE language is well-adapted to the simulation

- We focus mainly on BSDE:

$$Y_t = \Phi(X_T) + \int_t^T f(s, X_s, Y_s,)ds - \int_t^T Z_s dW_s,$$

  - where X is a forward SDE.
  - We know that $Y_t = u(t, X_t)$ and $Z_t = \nabla_x u(t, X_t)\sigma(t, X_t)$, where u solves a semi-linear PDE
- $\Longrightarrow$ to approximate (Y, Z), we need to approximate $u(\cdot)$, the gradiant of u and the process X
  - $Y_t^N = u^N(t, X_t^N)$,
  - in practice, X^N is always random,
  - although u is deterministic, u^N may be random (e.g. Monte Carlo approximations):
- Z is simulated using different methods used to approximate conditional expectation.

New Developments in BSDEs

Peng's G-expectation

- Motivation Volatility is uncertain, but preserved to lie in a fixed interval $D = [a, b]$.
- $\mathcal{E}^G((X)$ is the worst-case expectation of a random variable X over all these scenarios for the volatility.
- In the Markovian case, non-linear PDE BS-Blarenblatt (Avellaneda)

$$-u_t - G(u_{xx}) = 0, \quad u(T, x) = f(x), G(x) = \frac{1}{2} \sup_{y \in D}(xy)$$

- Extension to the canonical space with technical difficulties due to the loss of some reference probability
- The goal is to generalize to previous properties.

Correlated works

2BSDEs (Soner ,Touzi, Zhang, Nutz)

- Motivation Control on the delta BS

  $-dY_t = H_t(Y_t, Z_t, \Gamma_t)dt + Z_t.dW_t, \quad \Gamma_t dt = d < Z, W >$

Optimal transportation and Universal bounds Touzi, Tan, Labordere(SG)(2011)

- Superpricing of derivatives by general Itô semimartingales, with given marginal distributions = optimal transport
- Numerical methods

Functional Itô Calculus: Dupire (Bloombeg) Cont

- New notions of functional derivatives, well-adapted to functional hedging problems

Among Strategic Research Problem Post-Crisis

Global financial system

- Systemic risk and Instability:
- Collateralization, Impact and Sources of Risk
- Simulation of counterparty risk
- Impact of the regulation
- Transparency

Difficult to obtain data

Lack of information

Why these methods did not prevent the crisis?

- In the credit derivatives market, only too simplistic static models were used.
- In an incomplete market, it is difficult to estimate the residual risk.
- Daily risk management, by delta hedging or value at risk has to be completed by different indicators relative to different time scales
- Liquidity risk in particular was not captured
- Counterpart risk was minimized
- Systemic risk was undervalued
- Other indicators, such as the size of the positions (2000 Billion subprimes) exist outside of the mathematical criterion.

Demand for technology

- Wall Street is exploring the use of graphics processing units found in video games to speed up options analytic and other math-intensive applications
- All developments in Monte Carlo simulation are made efficient by the new power of the computer
- New developments in algorithmic trading, where an engine is used to place trades using an electronic order book

Markets are not like physical systems: Glenn

At least three common behaviors cannot be with (simple) maths:

- Intentionality of human actions/reactions,
- Subjective notion of risk,
- Strategic Behaviors,
- Asymmetric information.

So Game theory for example is to be taken into consideration, which is practically more difficult to deal with.

Conclusion

The end of a bubble between Mathematics and Finance: yes!

But not the end of mathematics in finance.

- Mathematicians bring rigor to the party, and rigor is a critical part of quantitative finance, and risk management.
- More demand for quantitative risk management.
- Technology evolves quickly in financial markets.

Still remember that in the social sciences, there are no true reproducible situations. So maths can only yield to partial representation of the complex reality.

Financial System in Equilibrium
Thank you for your attention